A MINIMUM VARIANCE DISTORTIONLESS RESPONSE SPECTRAL ESTIMATOR WITH KRONECKER PRODUCT FILTERS

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ABSTRACT

Spectral estimation is of significant practical importance in a wide range of applications. This paper proposes a minimum variance distortionless response (MVDR) method for spectral estimation based on the Kronecker product. Taking advantage of the particular structure of the Fourier vector, we decompose it as a Kronecker product of two shorter vectors. Then, we design the spectral estimation filters under the same structure, i.e., as a Kronecker product of two filters. Consequently, the conventional MVDR spectrum problem is transformed to one of estimating two filters of much shorter lengths. Since it has much fewer parameters to estimate, the proposed method is able to achieve better performance than its conventional counterpart, particularly when the number of available signal samples is small. Also presented in this paper is the generalization to the estimation of the cross-spectrum and coherence function.

Index Terms—Spectral estimation, cross-spectrum, coherence function, minimum variance distortionless response (MVDR) filter, Kronecker product.

1. INTRODUCTION

Spectral estimation, which aims at estimating the spectral density of a random signal from a sequence of observation samples, is widely used in applications such as speech analysis, radar, sonar, ultrasound, to name but a few [1–6]. Various spectrum estimation methods have been developed in the literature and the representative ones include the periodogram [7], Welch's method [8], minimum variance distortionless response (MVDR) technique [9], and the autoregressive moving average (ARMA) approach [10]. Generally, those methods can be broadly classified into two categories, i.e., non-parametric and parametric ones [11]. The former passes the broadband signal to be analyzed into a bank of band-pass filters and then computes the power of every subband filter's output [3, 12, 13]. In contrast, the latter assumes an *a priori* parametric model for the signal, and the spectral estimation problem is then transformed into one of estimating the parameters in the assumed model [11, 14]. Comparatively, the parametric methods can achieve more accurate estimation than the non-parametric ones if the signal fits well the assumed model. However, if the model is not a good representation of the signal, which is often true in most applications, those methods may suffer from great performance degradation. In summary, the non-parametric approaches are generally more robust than the parametric ones in practice and, therefore, plenty of efforts have been devoted to their study [3, 12–15].

One of the most well-known non-parametric algorithms is the Capon's approach, also known as the MVDR method. It achieves spectral estimation through a filterbank decomposition, where the spectrum of a given broadband signal is estimated on a subband basis [1, 9]. The resolution of this method depends on the length of the subband filters, and a higher resolution can be achieved by increasing the filter length but this will require more data samples (or signal snapshots) to estimate the signal covariance matrix of a larger size. In many practical applications, it is required to achieve reliable spectral estimates of high resolution with the constraint that the number of signal samples is limited at the given time instant [1, 2]. Therefore, efforts are indispensable to develop robust MVDR estimators using only a small number of samples.

The Kronecker product tool, which can decompose a long filter into several short ones, is very appealing for many applications [16, 17] such as system identification [18], beamforming [19–21], time-delay estimation [22], and dereverberation [23]. In this work, we attempt to apply this mathematical tool to robust spectral estimation and develop an MVDR method using Kronecker product filters. We first decompose the Fourier vector, which forms the filterbank in the MVDR, as a Kronecker product of two shorter vectors. As a result, the spectral estimation subband filter associated with the Fourier vector can also be expressed as a Kronecker product of two shorter filters. Consequently, the spectral estimation problem is transformed to one of estimating two optimal filters, which are achieved through an alternative least-squares (ALS) strategy. Since the number of parameters that needs to be estimated is significantly reduced with the Kronecker product structure, the proposed method can achieve better and robust spectral estimation performance than its conventional counterpart, particularly when the number of available samples is small. This proposed approach can also be generalized to the estimation of the cross-spectrum and coherence function, which is also addressed in this paper.

2. CONVENTIONAL MVDR SPECTRUM

The MVDR method performs spectral estimation on a subband basis, where in every subband an adaptive filter is designed to estimate the signal spectrum while minimizing the impact of signals from other subbands [1], [9]. Let us consider K complex-valued linear filters of length L:

$$\mathbf{g}_k = \begin{bmatrix} G_{k,0} & G_{k,1} & \cdots & G_{k,L-1} \end{bmatrix}^T, \quad k = 0, 1, \ldots, K-1, \tag{1}$$

where the superscript ^{T} is the transpose operator, and let $X(n)$ be a zero-mean stationary random process at the discrete-time index n, which is the input of these K filters. As a result, the corresponding outputs are

$$Y_k(n) = \mathbf{g}_k^H \mathbf{x}(n),\ k = 0, 1, \ldots, K-1, \tag{2}$$

where the superscript ^{H} is the conjugate-transpose operator and

$$\mathbf{x}(n) = \begin{bmatrix} X(n) & X(n-1) & \cdots & X(n-L+1) \end{bmatrix}^T$$

is a vector of length L containing the L most recent time samples of $X(n)$. Then, the variance of $Y_k(n)$ is

$$\phi_{Y_k} = \mathbf{g}_k^H \mathbf{R}_\mathbf{x} \mathbf{g}_k, \tag{3}$$

where $\mathbf{R}_\mathbf{x} = E\left[\mathbf{x}(n)\mathbf{x}^H(n)\right]$ is the covariance matrix of $\mathbf{x}(n)$, with $E[\cdot]$ denoting mathematical expectation. The variance ϕ_{Y_k}, with an appropriate constrained filter $\mathbf{g}_k$, is considered as the spectrum of $X(n)$ at frequency bin k that we denote by $\mathcal{S}_X(\omega_k)$, where $\omega_k = 2\pi k/K$.

Consider the matrix of size $L \times K$:

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}_0 & \mathbf{f}_1 & \cdots & \mathbf{f}_{K-1} \end{bmatrix}, \tag{4}$$

where

$$\mathbf{f}_k = \frac{1}{\sqrt{L}} \begin{bmatrix} 1 & e^{\jmath\omega_k} & \cdots & e^{\jmath\omega_k(L-1)} \end{bmatrix}^T \tag{5}$$

is the Fourier vector of length L, with $\jmath$ being the imaginary unit. For $K = L$, $\mathbf{F}$ is the well-known Fourier matrix, which is unitary, i.e., $\mathbf{F}^H\mathbf{F} = \mathbf{F}\mathbf{F}^H = \mathbf{I}_L$, where $\mathbf{I}_L$ is the $L \times L$ identity matrix. In the MVDR spectrum, the filter coefficients are chosen so as to minimize the variance of the filter output subject to the constraint:

$$\mathbf{g}_k^H \mathbf{f}_k = \mathbf{f}_k^H \mathbf{g}_k = 1. \tag{6}$$

Under this constraint, the process $X(n)$ is passed through the filter $\mathbf{g}_k$ with no distortion at frequency ω_k and components at other frequencies than ω_k tend to be attenuated. Mathematically, this is equivalent to

$$\min_{\mathbf{g}_k} \left(\phi_{Y_k} = \mathbf{g}_k^H \mathbf{R}_\mathbf{x} \mathbf{g}_k \right) \quad \text{s. t.} \quad \mathbf{g}_k^H \mathbf{f}_k = 1. \tag{7}$$

We easily find that the optimal filter is

$$\mathbf{g}_k = \frac{\mathbf{R}_\mathbf{x}^{-1}\mathbf{f}_k}{\mathbf{f}_k^H \mathbf{R}_\mathbf{x}^{-1} \mathbf{f}_k}. \tag{8}$$

As a consequence, the spectrum of $X(n)$ at ω_k is

$$\mathcal{S}_X(\omega_k) = \mathbf{g}_k^H \mathbf{R}_\mathbf{x} \mathbf{g}_k = \frac{1}{\mathbf{f}_k^H \mathbf{R}_\mathbf{x}^{-1} \mathbf{f}_k}. \tag{9}$$

While it gives pretty satisfactory spectral estimation in practice with a high resolution by simply increasing the value of L, the MVDR approach is required to have a large amount of data in order to obtain a good estimate of $\mathbf{R}_\mathbf{x}$ so that its inversion does not lead to potential problems. In many applications, however, we rarely are in a such luxury context. Therefore, there is a great need to estimate related covariance matrices with much less data.

3. PROPOSED APPROACH

3.1. Spectral Estimation

Let us assume that $L = L_1 \times L_2$. In this case, one can easily check that $\mathbf{f}_k$ can be decomposed as

$$\mathbf{f}_k = \mathbf{f}_{[1],k} \otimes \mathbf{f}_{[2],k}, \tag{10}$$

where $\otimes$ is the Kronecker product and

$$\mathbf{f}_{[1],k} = \frac{1}{\sqrt{L_1}} \begin{bmatrix} 1 & e^{\jmath\omega_k L_2} & \ldots & e^{\jmath\omega_k L_2(L_1-1)} \end{bmatrix}^T, \tag{11}$$

$$\mathbf{f}_{[2],k} = \frac{1}{\sqrt{L_2}} \begin{bmatrix} 1 & e^{\jmath\omega_k} & \ldots & e^{\jmath\omega_k(L_2-1)} \end{bmatrix}^T \tag{12}$$

are vectors of lengths L_1 and L_2, respectively.

Because of the particular structure of $\mathbf{f}_k$, we propose to use the same structure for the K linear global filters, i.e.,

$$\mathbf{g}_k = \mathbf{g}_{[1],k} \otimes \mathbf{g}_{[2],k}, \tag{13}$$

where $\mathbf{g}_{[1],k}$ and $\mathbf{g}_{[2],k}$ are two complex-valued sub-filters of lengths L_1 and L_2, respectively. We can rewrite (13) as

$$\mathbf{g}_k = \left(\mathbf{I}_{L_1} \otimes \mathbf{g}_{[2],k}\right) \mathbf{g}_{[1],k} \tag{14}$$

$$= \left(\mathbf{g}_{[1],k} \otimes \mathbf{I}_{L_2}\right) \mathbf{g}_{[2],k}, \tag{15}$$

where $\mathbf{I}_{L_1}$ and $\mathbf{I}_{L_2}$ are the identity matrices of sizes $L_1 \times L_1$ and $L_2 \times L_2$, respectively.

In the context of the proposed approach, the constraint on the global filter in (6) becomes

$$\left(\mathbf{g}_{[1],k} \otimes \mathbf{g}_{[2],k}\right)^H \left(\mathbf{f}_{[1],k} \otimes \mathbf{f}_{[2],k}\right) = \mathbf{g}_{[1],k}^H \mathbf{f}_{[1],k} \times \mathbf{g}_{[2],k}^H \mathbf{f}_{[2],k} = 1. \tag{16}$$

Therefore, we will always take $\mathbf{g}_{[1],k}^H \mathbf{f}_{[1],k} = \mathbf{g}_{[2],k}^H \mathbf{f}_{[2],k} = 1$, so that (16) is satisfied.

Closed-form expressions for the two sub-filters $\mathbf{g}_{[1],k}$ and $\mathbf{g}_{[2],k}$ do not exist but good approximations can be derived thanks to the alternative least-squares (ALS) strategy [16, 19]. When $\mathbf{g}_{[2],k}$ is fixed, we write the variance of $Y_k(n)$ as

$$\phi_{Y_{\mathbf{g}_{[2],k}}} = \mathbf{g}_{[1],k}^H \left(\mathbf{I}_{L_1} \otimes \mathbf{g}_{[2],k}\right)^H \mathbf{R}_\mathbf{x} \left(\mathbf{I}_{L_1} \otimes \mathbf{g}_{[2],k}\right) \mathbf{g}_{[1],k} = \mathbf{g}_{[1],k}^H \mathbf{R}_{\mathbf{x},\mathbf{g}_{[2],k}} \mathbf{g}_{[1],k}^H, \tag{17}$$

where

$$\mathbf{R}_{\mathbf{x},\mathbf{g}_{[2],k}} = \left(\mathbf{I}_{L_1} \otimes \mathbf{g}_{[2],k}\right)^H \mathbf{R}_\mathbf{x} \left(\mathbf{I}_{L_1} \otimes \mathbf{g}_{[2],k}\right) \tag{18}$$

is a covariance matrix of size $L_1 \times L_1$; and when $\mathbf{g}_{[1],k}$ is fixed, we write the variance of $Y_k(n)$ as

$$\phi_{Y_{\mathbf{g}_{[1],k}}} = \mathbf{g}_{[2],k}^H \left(\mathbf{g}_{[1],k} \otimes \mathbf{I}_{L_2}\right)^H \mathbf{R}_\mathbf{x} \left(\mathbf{g}_{[1],k} \otimes \mathbf{I}_{L_2}\right) \mathbf{g}_{[2],k} = \mathbf{g}_{[2],k}^H \mathbf{R}_{\mathbf{x},\mathbf{g}_{[1],k}} \mathbf{g}_{[2],k}^H, \tag{19}$$

where

$$\mathbf{R}_{\mathbf{x},\mathbf{g}_{[1],k}} = \left(\mathbf{g}_{[1],k} \otimes \mathbf{I}_{L_2}\right)^H \mathbf{R}_\mathbf{x} \left(\mathbf{g}_{[1],k} \otimes \mathbf{I}_{L_2}\right) \tag{20}$$

is a covariance matrix of size $L_2 \times L_2$. Therefore, the optimal global filter is derived iteratively from

$$\min_{\mathbf{g}_{[1],k}} \left(\phi_{Y_{\mathbf{g}_{[2],k}}} = \mathbf{g}_{[1],k}^H \mathbf{R}_{\mathbf{x},\mathbf{g}_{[2],k}} \mathbf{g}_{[1],k}\right) \quad \text{s. t.} \quad \mathbf{g}_{[1],k}^H \mathbf{f}_{[1],k} = 1, \tag{21}$$

$$\min_{\mathbf{g}_{[2],k}} \left(\phi_{Y_{\mathbf{g}_{[1],k}}} = \mathbf{g}_{[2],k}^H \mathbf{R}_{\mathbf{x},\mathbf{g}_{[1],k}} \mathbf{g}_{[2],k}\right) \quad \text{s. t.} \quad \mathbf{g}_{[2],k}^H \mathbf{f}_{[2],k} = 1, \tag{22}$$

whose solutions are

$$\mathbf{g}_{[1],k} = \frac{\mathbf{R}_{\mathbf{x},\mathbf{g}_{[2],k}}^{-1} \mathbf{f}_{[1],k}}{\mathbf{f}_{[1],k}^H \mathbf{R}_{\mathbf{x},\mathbf{g}_{[2],k}}^{-1} \mathbf{f}_{[1],k}}, \tag{23}$$

$$\mathbf{g}_{[2],k} = \frac{\mathbf{R}_{\mathbf{x},\mathbf{g}_{[1],k}}^{-1} \mathbf{f}_{[2],k}}{\mathbf{f}_{[2],k}^H \mathbf{R}_{\mathbf{x},\mathbf{g}_{[1],k}}^{-1} \mathbf{f}_{[2],k}}. \tag{24}$$

As a result, at iteration i, the global MVDR filter is

$$\mathbf{g}_k^{(i)} = \mathbf{g}_{[1],k}^{(i)} \otimes \mathbf{g}_{[2],k}^{(i)}, \tag{25}$$

where

$$\mathbf{g}_{[1],k}^{(i)} = \frac{\left(\mathbf{R}_{\mathbf{x},\mathbf{g}_{[2],k}}^{(i)}\right)^{-1} \mathbf{f}_{[1],k}}{\mathbf{f}_{[1],k}^H \left(\mathbf{R}_{\mathbf{x},\mathbf{g}_{[2],k}}^{(i)}\right)^{-1} \mathbf{f}_{[1],k}}, \tag{26}$$

$$\mathbf{g}_{[2],k}^{(i)} = \frac{\left(\mathbf{R}_{\mathbf{x},\mathbf{g}_{[1],k}}^{(i)}\right)^{-1} \mathbf{f}_{[2],k}}{\mathbf{f}_{[2],k}^H \left(\mathbf{R}_{\mathbf{x},\mathbf{g}_{[1],k}}^{(i)}\right)^{-1} \mathbf{f}_{[2],k}}, \tag{27}$$

and the iteratively-updated covariance matrices are given by

$$\mathbf{R}_{\mathbf{x},\mathbf{g}_{[2],k}}^{(i)} = \left(\mathbf{I}_{L_1} \otimes \mathbf{g}_{[2],k}^{(i-1)}\right)^H \mathbf{R}_{\mathbf{x}} \left(\mathbf{I}_{L_1} \otimes \mathbf{g}_{[2],k}^{(i-1)}\right), \tag{28}$$

$$\mathbf{R}_{\mathbf{x},\mathbf{g}_{[1],k}}^{(i)} = \left(\mathbf{g}_{[1],k}^{(i)} \otimes \mathbf{I}_{L_2}\right)^H \mathbf{R}_{\mathbf{x}} \left(\mathbf{g}_{[1],k}^{(i)} \otimes \mathbf{I}_{L_2}\right). \tag{29}$$

Since the covariance matrices $\mathbf{R}_{\mathbf{x},\mathbf{g}_{[2],k}}$ and $\mathbf{R}_{\mathbf{x},\mathbf{g}_{[1],k}}$ are only of sizes $L_1 \times L_1$ and $L_2 \times L_2$, respectively, they require much less data to estimate than the original covariance matrix $\mathbf{R}_{\mathbf{x}}$ (of size $L_1 L_2 \times L_1 L_2$).

The spectrum of $X(n)$ at ω_k (and iteration i) with the global MVDR filter is

$$\mathcal{S}_X^{(i)}(\omega_k) = \frac{1}{\mathbf{f}_{[1],k}^H \left(\mathbf{R}_{\mathbf{x},\mathbf{g}_{[2],k}}^{(i)}\right)^{-1} \mathbf{f}_{[1],k}} = \frac{1}{\mathbf{f}_{[2],k}^H \left(\mathbf{R}_{\mathbf{x},\mathbf{g}_{[1],k}}^{(i)}\right)^{-1} \mathbf{f}_{[2],k}}. \tag{30}$$

3.2. Estimation of the Cross-Spectrum and Coherence Function

The method developed in the previous subsection can also be extended to the estimation of the cross-spectrum and coherence function. Let us assume that we have two zero-mean stationary random signals $X_1(n)$ and $X_2(n)$ with respective spectra $\mathcal{S}_{X_1}(\omega_k)$ and $\mathcal{S}_{X_2}(\omega_k)$. As explained above, we can design two global MVDR filters:

$$\mathbf{g}_{p,k}^{(i)} = \mathbf{g}_{p,[1],k}^{(i)} \otimes \mathbf{g}_{p,[2],k}^{(i)}, \ p = 1, 2, \tag{31}$$

where $\mathbf{g}_{1,k}^{(i)}$ (resp. $\mathbf{g}_{2,k}^{(i)}$) of length L is used for the estimation of the spectrum of $X_1(n)$ [resp. $X_2(n)$], i.e., $\mathcal{S}_{X_1}^{(i)}(\omega_k)$ [resp. $\mathcal{S}_{X_2}^{(i)}(\omega_k)$], at ω_k (and iteration i).

Let $Y_{1,k}(n)$ and $Y_{2,k}(n)$ be the respective outputs of the filters $\mathbf{g}_{1,k}$ and $\mathbf{g}_{2,k}$ of length L. With the conventional approach, the cross-spectrum between $X_1(n)$ and $X_2(n)$ at frequency ω_k is defined as [11], [14]

$$\mathcal{S}_{X_1 X_2}(\omega_k) = E\left[Y_{1,k}(n) Y_{2,k}^*(n)\right], \tag{32}$$

where the superscript * is the complex-conjugate operator. Developing the previous expression, we get

$$\mathcal{S}_{X_1 X_2}(\omega_k) = \mathbf{g}_{1,k}^H \mathbf{R}_{\mathbf{x}_1 \mathbf{x}_2} \mathbf{g}_{2,k}, \tag{33}$$

where $\mathbf{R}_{\mathbf{x}_1 \mathbf{x}_2} = E\left[\mathbf{x}_1(n)\mathbf{x}_2^H(n)\right]$ is the $L \times L$ cross-correlation matrix between the two vectors $\mathbf{x}_1(n)$ and $\mathbf{x}_2(n)$ of length L, which are defined similarly to $\mathbf{x}(n)$. Therefore, with our approach, the cross-spectrum between $X_1(n)$ and $X_2(n)$ at frequency ω_k (and iteration i) is given by

$$\mathcal{S}_{X_1 X_2}^{(i)}(\omega_k) = \left(\mathbf{g}_{1,k}^{(i)}\right)^H \mathbf{R}_{\mathbf{x}_1 \mathbf{x}_2} \mathbf{g}_{2,k}^{(i)}. \tag{34}$$

The coherence function between the two signals $X_1(n)$ and $X_2(n)$ is defined as [11], [14]

$$\gamma_{X_1 X_2}(\omega_k) = \frac{\mathcal{S}_{X_1 X_2}(\omega_k)}{\sqrt{\mathcal{S}_{X_1}(\omega_k)\,\mathcal{S}_{X_2}(\omega_k)}}. \tag{35}$$

Therefore, with the proposed approach, this coherence function is estimated as follows:

$$\gamma_{X_1 X_2}^{(i)}(\omega_k) = \frac{\mathcal{S}_{X_1 X_2}^{(i)}(\omega_k)}{\sqrt{\mathcal{S}_{X_1}^{(i)}(\omega_k)\,\mathcal{S}_{X_2}^{(i)}(\omega_k)}} = \frac{\left(\mathbf{g}_{1,k}^{(i)}\right)^H \mathbf{R}_{\mathbf{x}_1 \mathbf{x}_2} \mathbf{g}_{2,k}^{(i)}}{\sqrt{\left(\mathbf{g}_{1,k}^{(i)}\right)^H \mathbf{R}_{\mathbf{x}_1} \mathbf{g}_{1,k}^{(i)} \times \left(\mathbf{g}_{2,k}^{(i)}\right)^H \mathbf{R}_{\mathbf{x}_2} \mathbf{g}_{2,k}^{(i)}}}, \tag{36}$$

where $\mathbf{R}_{\mathbf{x}_1}$ and $\mathbf{R}_{\mathbf{x}_2}$ are the covariance matrices of $\mathbf{x}_1(n)$ and $\mathbf{x}_2(n)$, respectively.

4. SIMULATIONS

In this section, we study the performance of the proposed method and compare it to the conventional MVDR method. We consider a synthetic signal, which consists of multiple sinusoids, i.e.,

$$X(n) = 4\cos(0.42\pi n) + 4\cos(0.44\pi n) + 6\cos(0.46\pi n) + 6\cos(0.47\pi n) + V_1(n), \tag{37}$$

where $V_1(n)$ is a zero-mean white gaussian noise. To estimate the spectrum of the above signal, the number of the complex-valued linear filters, i.e., the value of K, is set to $K = 1000$. The length of each filter, i.e., the value of L, is set to $L = 100$. The proposed method is implemented with $L_1 = L_2 = 10$. The filter $\mathbf{g}_{[2],k}$ is initialized as $\mathbf{f}_{[2],k}$, and only two iterations are performed, i.e., $I = 2$.

For comparison, we also show performance of the Welch average periodogram method. The covariance matrix $\mathbf{R}_{\mathbf{x}}$ is estimated with the forward and backward method [24, 25]. The estimates given by the MVDR and the proposed method are divided

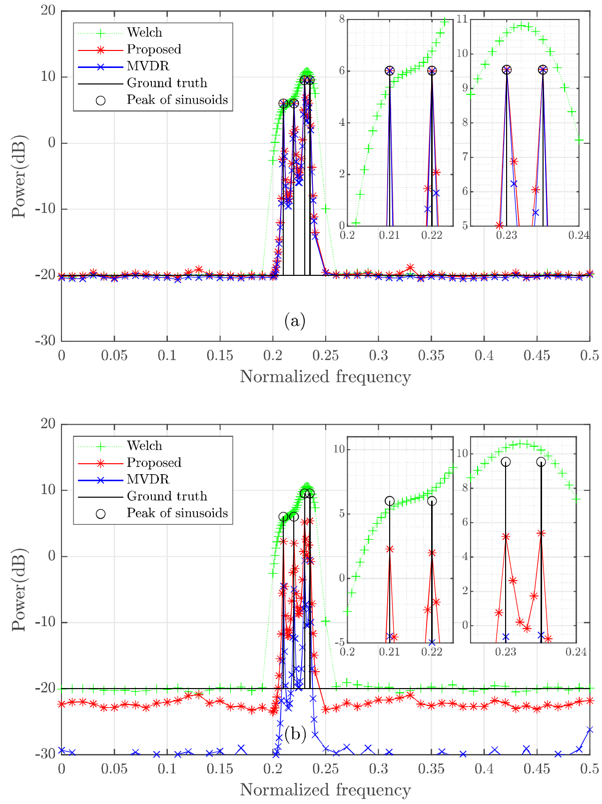

Fig. 1. Spectrum estimates with the Welch average periodogram, MVDR, and proposed methods with: (a) 150 snapshots and (b) 500 snapshots. The results are from averaging over 100 Monte Carlo trails.

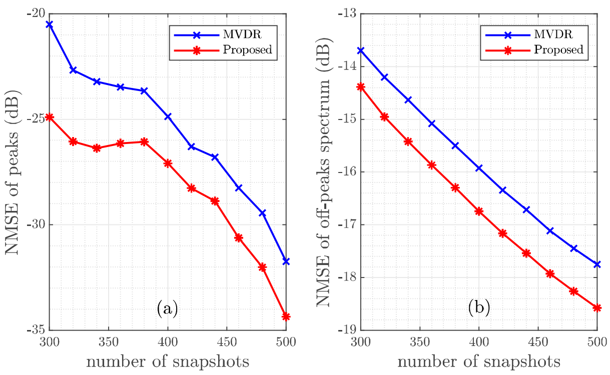

Fig. 2. MSC estimation performance of the MVDR and proposed methods with difference number of snapshots: (a) NMSE of peaks and (b) NMSE of off-peaks spectrum. The results are averaged over 1000 Monte Carlo trails.

by L. For the MVDR method, the diagonal loading factor is set as $\delta = \text{trace}(\mathbf{R_x}) \times 10^{-6}/L$ [26]. We consider two cases, where the number of available snapshots is 150 and 500, respectively. Figure 1 shows plots of the averaged results over 100 Monte Carlo simulations. As seen, the Welch average periodogram method has a much lower resolution than the other two studied methods in all cases. When $N = 500$, both MVDR and the proposed method produce good performance and the estimated spectra are close to the ground truth. The underlying reason is that, with $N = 500$, the covariance matrix is well estimated. However, when $N = 150$, both methods suffer great performance degradation because the estimate of the covariance matrix is biased. In this case, the amplitude of the spectral estimate with MVDR is much smaller than the ground truth. In comparison, the proposed method has better overall performance, which achieves both comparable estimation accuracy and high resolution. The reason, as explained in Section 3, is that the dimensions of the $\mathbf{R}_{\mathbf{x},\mathbf{g}_{[1],k}}$ and $\mathbf{R}_{\mathbf{x},\mathbf{g}_{[2],k}}$ matrices are much smaller than that of $\mathbf{R_x}$; consequently, less data are needed to achieve reliable estimates.

We then compare the magnitude squared coherence (MSC) estimation performance of the proposed method with MVDR with different number of snapshots. We consider the synthetic signal:

$$\begin{aligned} Y(n) &= \cos(0.42\pi n + \phi_1) + \cos(0.44\pi n + \phi_2) \\ &+ \cos(0.46\pi n + \phi_3) + \cos(0.48\pi n + \phi_4) + V_2(n), \end{aligned} \tag{38}$$

where $\phi_k \in [0, 2\pi)$ is a random phase and $V_2(n)$ is a zero-mean white gaussian noise with variance $\sigma^2 = 0.01$. The MSC between $X(n)$ and $Y(n)$ is expected to be 1 at $\omega = 0.42\pi, 0.44\pi, 0.46\pi$ and 0 at other frequencies. We consider the normalized mean-squared error (NMSE) of those three peaks and the NMSE of off-peak spectrum. The number of available snapshots varies from 300 to 500 with an increment of 20. Figure 2 shows plots of the averaged results over 1000 Monte Carlo simulations. As seen, the performance of both algorithms decrease as the number of available snapshots decreases. In comparison, the proposed method yields both more accurate peak height estimation and lower off-peaks spectrum power estimation, indicating that the proposed MSC estimator outperforms the conventional MVDR.

5. CONCLUSIONS

This paper deals with the problem of spectral estimation. It presented an MVDR method based on the Kronecker product decomposition, where the spectral estimation filter is expressed as a Kronecker product of two short filters, which are estimated with the ALS method. Since it has much fewer parameters to estimate, the developed method is able to achieve better performance than its conventional counterpart when the number of available signal samples is small, which was justified by both theoretical analysis as well as simulation results. We also showed how to generalize the proposed spectral estimation method to the estimation of cross-spectrum and coherence function.

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